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1997 J. Phys. A: Math. Gen. 30 7639

(http://iopscience.iop.org/0305-4470/30/21/031)

J. Phys. A: Math. Gen. **30** (1997) 7639–7658. Printed in the UK PII: S0305-4470(97)77516-4

Similarity solution for fragmentation with volume change

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Received 28 August 1996, in final form 16 June 1997

Abstract. The similarity form of the solution of the linear homogeneous equation for fragmentation with volume change is given. The solution shows the time evolution of the particle-volume distribution when a particle splits into a polynomial distribution of fragment volumes. The limits for small and large values of the similarity variable are derived, the long-time limit for all values of the similarity variable is given and the characteristic time to approach the limit is identified. The solution shows the effect of volume change on the particle-volume distribution and on the time dependence of the moments of the distribution.

1. Introduction

A continuous linear equation called the fragmentation equation was introduced by Fillipov [1] as a model for the fragmentation (splitting) of particles. Since then the solutions of the equation have been studied by many authors where [1–15] are relevant to the analysis given here. The equation determines the evolution in time of the volume distribution of particles as fragmentation into smaller particles proceeds. The assumptions for the theory here and in the above references are: (i) the particles are characterized by only one variable such as particle mass or a characteristic dimension of the particle, which here we call *particle volume*; (ii) the fragmentation of the particles does not require interaction with other particles of the same kind and in this sense is spontaneous. Consequently the fragmentation equation is linear; (iii) the discrete nature of the particles is ignored by taking the distribution of the particles to be a function of a continuous particle-volume variable x, where $x \geqslant 0$ and thus there is no small particle fragmentation cut-off; (iv) the probability of a particle fragmenting is assumed to be independent of its history and proportional to x^{α} where $\alpha > 0$ is a real parameter (the assumption $\alpha > 0$ is necessary for the existence of a similarity solution and excludes the shattering kind of fragmentation); (v) changes in the particle-volume distribution caused by spatial gradients in the concentration of particles are assumed negligible, so spatial coordinates are absent from the theory. Later we will make a further assumption for the function form of the probability for the number and distribution of fragment volumes when a particle splits into fragments.

We refer to Edwards *et al* [6] and Huang *et al* [10] for a discussion of physical fragmentation processes in combustion where particle volume (or mass) can change. For further applications we refer to Ziff and McGrady [4] on the fragmentation of polymers and to Redner [9] for a general discussion of fragmentation processes. The emphasis here is on the mathematics of constructing an exact solution for fragmentation with volume change where we point out properties of the solution that should be of interest for physical applications.

The fragmentation equation provides an accounting in time of the number and volumes of the particles as they fragment into smaller particles. The equation assumed here and in [1–15] is

$$\frac{\partial}{\partial t} n(x,t) = -\tilde{c}_\alpha x^\alpha n(x,t) + \tilde{c}_\alpha \int_x^\infty y^{\alpha-1} b(x/y) n(y,t)\,\mathrm{d}y. \tag{1.1}$$

The notation and the description of the terms are the same as in [15] but for convenience we repeat them here. In (1.1) x is the volume of a particle with the dimension of length cubed, α is the degree of homogeneity in x, t is time in seconds and $n(x,t)$ is the uniform spatial concentration of particles per unit particle volume with dimension x^{-2}. Throughout the analysis $\alpha > 0$, which is a necessary condition for the existence of a similarity solution. The first term on the right-hand side of (1.1) is the rate at which the concentration of particles of volume x decreases by fragmentation and the second term is the rate at which the concentration of particles of volume x increases due to the fragmentation of particles with volumes larger than x. The rate constant $\tilde{c}_\alpha$ has the dimensions of $x^{-\alpha}t^{-1}$. The daughter-fragment distribution, $b(x/y)$, determines the number and distribution of fragment volumes when a particle splits into fragments. We assume that $b(x/y)$ is a function of the ratio x/y and thus is homogeneous of degree zero. The fragmentation terms are, therefore, homogeneous of degree α which is necessary for the existence of solutions of the similarity form. If $\int_0^1 rb(r)\,\mathrm{d}r = 1$ then the first moment of the particle distribution, which is the volume (or mass), *is conserved* in the fragmentation process. However, if $\int_0^1 rb(r)\,\mathrm{d}r \neq 1$ then the volume *is not conserved*. Fillipov [1], Edwards *et al* [6], Cai *et al* [7] and Huang *et al* [10] have allowed this generality in the theory. The daughter distribution determines the number of fragments per fragmentation, which we call *fragment number* and is given by $\tilde{N} = \int_0^1 b(r)\,\mathrm{d}r$. The fragment number does not change during the evolution of the distribution so as the particles become smaller they continue to split into the same number and the same distribution of fragments.

Edwards *et al* [6], Cai *et al* [7] and Huang *et al* [10] characterize the change in volume that occurs in the fragmentation process as *discrete*. The same authors also consider another kind of change in particle volume caused by a surface effect, such as evaporation or condensation, that changes the particle volume without changing the number of particles. Here we consider volume change by fragmentation and *do not* consider volume change by surface effects.

In [15] a similarity solution for *volume-conserving fragmentation* for a polynomial daughter distribution has been given. We show here that the solution given in [15] generalizes in a simple way to give the solution for *fragmentation with volume change*. Unfortunately the similarity solution is not the general solution of the initial value problem. However, Kolmogorov [16] has shown in a special case and Fillipov [1] has shown for conditions satisfied here (except for volume conservation) that for long times the general solution converges to a limit distribution. The theorem should also hold for fragmentation with volume change. The limit distribution is determined here by taking the long-time limit of the similarity solution (called a stationary solution by Fillipov) and thus, after a sufficiently long time, other solutions with initial conditions different than for the similarity solution should approach the limit of the similarity solution.

We recall that in the theory of volume-conserving *coagulation* the Friedlander ansatz [17] assumes that the solution for the particle-volume distribution has the similarity form

$$n(x,t) = \frac{N(t)\phi(z)}{V} \qquad z = \frac{N(t)x}{V} \tag{1.2}$$

where $\phi(z)$ is called the reduced distribution, $N(t)$ is the total number of particles and V is the conserved total particle volume. The similarity variable z is the ratio of the volume x of an individual particle to the instantaneous mean particle volume $v(t) = V/N(t)$, where $N(t)$ is the zeroth moment and V is the first moment of the distribution. The kth moment of the distribution is defined by

$$M_k = \int_0^\infty x^k n(x,t)\,\mathrm{d}x \tag{1.3}$$

with k a real number. As shown, for example, in [17], substitution of the Friedlander similarity form into the volume-conserving *coagulation* equation separates the partial derivative equation into an ordinary equation in time for the moments M_k and an ordinary equation in the similarity variable for the reduced distribution $\phi(z)$. This provides the mathematical advantage of working with two ordinary equations rather than a partial differential equation. Likewise, for volume-conserving *fragmentation*, substitution of the Friedlander form into (1.1) separates the equation in essentially the same way as for coagulation. The Friedlander ansatz separates variables for both the *nonlinear coagulation equation* and the *linear fragmentation equation* because both equations are homogeneous in the particle volume and are first order in the time derivative and thus have the same scaling invariance.

To obtain the similarity form of solution for fragmentation which *does not conserve volume* one needs to generalize the Friedlander similarity form (1.2). We obtain the generalization by following closely the analysis of the scaling invariance given in [15]. Then, as with volume-conserving fragmentation, substitution of the generalized Friedlander form into the fragmentation equation (1.1) separates the equation into ordinary equations for the moments and the reduced distribution. We then proceed with the solution of the moment equation and the reduced equation by the same steps taken in [15]. The solution of the reduced equation was obtained in [15] by the Mellin transformation used earlier by Ziff and McGrady [4], Cheng and Redner [5, 8] and Ziff [11]. The solution $n(x,t)$ in the form given by (1.2) for volume-conserving fragmentation was found to be proportional to a Meijer G-function which may be expressed as a linear combination of generalized hypergeometric functions. We show here that the similarity solution for fragmentation with volume change is a simple generalization of the solution for volume-conserving fragmentation. The solution for small z is expressed as a sum of generalized hypergeometric series in powers of z^α. The continuation of the small-z solution to large values of z is an *asymptotically converging* series in powers of $z^{-\alpha}$.

The small- and large-z limits of the solution are shown for all times and the long-time limit is shown for all z. We discuss the change of the shape of the distribution as the fragmentation proceeds and identify the characteristic time to reach the limit distribution and the characteristic times to change the moments of the distribution. Also, we give two simple examples of the similarity solution which illustrate how the allowed numerical ranges of the fragmentation parameters are determined and show in a simple setting the small- and large-z limits and the long-time behaviour of the solution.

Finally, in the summary and discussion section we collect and comment on the results of the analysis that seem most significant. In the appendices we give a derivation of the generalization of the Friedlander similarity form for fragmentation with volume change and we show some of the details of the special function analysis.

2. The moment equation when there is volume change

We first consider the moments of the distribution in a general way that does not involve a Friedlander ansatz or a similarity solution. Taking moments (the Mellin transform) of (1.1) we obtain

$$\frac{\mathrm{d}M_k(t)}{\mathrm{d}t} = -\tilde{c}_\alpha\left(1 - \int_0^1 r^k b(r)\,\mathrm{d}r\right)M_{k+\alpha}(t). \tag{2.1}$$

We suppose that the daughter distribution is such that there is a positive value $k = \lambda$ for which

$$\int_0^1 r^\lambda b(r)\,\mathrm{d}r = 1. \tag{2.2}$$

If (2.2) is satisfied it follows from (2.1) that the moment M_λ is constant. This is an important assumption that will be satisfied by the function $b(r)$ that we consider later. As a simple example of (2.2) let $b(r) = b_0 r^\gamma$ where b_0 and γ are real numbers. Then,

$$\int_0^1 r^\lambda b(r)\,\mathrm{d}r = b_0/(\lambda + \gamma + 1) = 1$$

and solving for λ we have $\lambda = b_0 - \gamma - 1$. If $b_0 = \gamma + 2$ then $\lambda = 1$ and volume is conserved. The daughter distribution $b(r) = b_0 r^\gamma$ with $b_0 = \gamma + 2$ has been considered in the papers [1–15]. However, if $b_0 \neq \gamma + 2$ then $\lambda \neq 1$ and volume is not conserved but the moment M_λ is conserved, which is a generalization that has been considered in [1, 6, 7, 10, 14]. We suppose for a general $b(r)$ that (2.2) is satisfied and thus the moment M_λ is constant. Since the coefficient of $M_{k+\alpha}$ in the moment equation vanishes for $k = \lambda$, we may write the coefficient in (2.1) in the factored form

$$\frac{\mathrm{d}M_k(t)}{\mathrm{d}t} = -\tilde{c}_\alpha(k - \lambda)A_k M_{k+\alpha}(t) \tag{2.3}$$

where A_k is a function of k defined by

$$(k - \lambda)A_k = 1 - \int_0^1 r^k b(r)\,\mathrm{d}r. \tag{2.4}$$

We impose the additional restriction on the daughter-fragment distribution that it is such that $A_k > 0$ for $k \geqslant \lambda - 1$.

Since the particle distribution $n(x, t)$ is non-negative it follows that the moments M_k defined by (1.3) are positive. Thus, by inspection of (2.3), we see that the moments $k < \lambda$ increase and the moments $k > \lambda$ decrease in time. In particular, for $k = 0$ and $k = 1$ we have

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \tilde{c}_\alpha(\tilde{N} - 1)M_\alpha(t) \qquad \frac{\mathrm{d}V(t)}{\mathrm{d}t} = -\tilde{c}_\alpha(1 - B)M_{1+\alpha}(t) \tag{2.5}$$

where $N(t)$ is the total number of particles, $V(t)$ is the total volume of particles and

$$\tilde{N} = \int_0^1 b(r)\,\mathrm{d}r \qquad B = \int_0^1 r b(r)\,\mathrm{d}r. \tag{2.6}$$

It follows from (2.5) that $\tilde{N}$ is the number of fragments per fragmentation and that if $B = 1$ the volume of particles is conserved in the fragmentation process, as was asserted in the introduction.

3. The generalization of the Friedlander similarity form and the similarity solution of the moment equation

The similarity ansatz given by equation (1.2) was first applied in volume-conserving *coagulation* theory by Friedlander [17] and was first applied in volume-conserving *fragmentation* theory by Peterson [3]. Since then the similarity form of the distribution or a limiting form of it have been assumed in a number of studies of the fragmentation equation. In [15], rather than introduce the similarity solution as an ansatz it was deduced as a consequence of the particle volume and time scaling invariance of the fragmentation equation. Following closely the derivation given in [15], where the volume is constant, we have derived a generalization of the Freidlander function form when a moment M_λ, $\lambda \neq 1$, instead of the volume $V = M_1$ is conserved. We show in appendix A that the generalization is

$$n(x,t) = \frac{M_{\lambda-1}^{\lambda+1}(t)\phi(z)}{M_\lambda^\lambda} \qquad z = \frac{M_{\lambda-1}(t)x}{M_\lambda} \tag{3.1}$$

where the upper indices are powers of the moments and the lower indices identify the moment. By comparing (3.1) with the volume-conserving similarity form (1.2) we can understand this combination of variables without going through the derivation. First we see that for $\lambda \to 1$ we recover the Friedlander form. Then we see that in the similarity variable (still called z) the volume V has been replaced by the conserved moment M_λ. In the numerator the particle number $N(t) = M_0(t)$ has been replaced by the moment $M_{\lambda-1}(t)$, with the index shifted by one from the conserved moment to make the similarity variable dimensionless. Likewise for the time-dependent multiplicative factor, the volume in the denominator has been replaced by the conserved moment M_λ and the particle number $N(t)$ in the numerator has been replaced by $M_{\lambda-1}(t)$. Then the powers are chosen so that $n(x,t)$ has the correct dimension x^{-2}. The λ-moment of (3.1) is

$$\int_0^\infty x^\lambda n(x,t)\,\mathrm{d}x = \int_0^\infty x^\lambda \frac{M_{\lambda-1}^{\lambda+1}(t)\phi(z)}{M_\lambda^\lambda}\,\mathrm{d}x = M_\lambda \int_0^\infty z^\lambda \phi(z)\,\mathrm{d}z$$

which confirms that the moment M_λ is identically constant for a solution of the form (3.1) and incidentally implies the normalization $\int_0^\infty z^\lambda \phi(z)\,\mathrm{d}z = 1$. We say that a solution is a *similarity solution* if it has the form (3.1) and satisfies the fragmentation equation (1.1) for $t \geqslant 0$, where the moment $M_{\lambda-1}$ and all higher moments are finite.

We first derive the similarity solution of the moment equation and then go on to derive the solution for the reduced distribution $\phi(z)$. To derive the moment solution we substitute the general similarity form (3.1) into the definition of the moments (1.3) and obtain

$$M_k(t) = \mu_k \frac{M_\lambda^{k+1-\lambda}}{M_{\lambda-1}^{k-\lambda}(t)} \tag{3.2}$$

where

$$\mu_k = \int_0^\infty z^k \phi(z)\,\mathrm{d}z \tag{3.3}$$

are the *reduced moments*. In particular, for $k = \lambda$ we have $M_\lambda = \mu_\lambda M_\lambda$ and for $k = \lambda - 1$ we have $M_{\lambda-1} = \mu_{\lambda-1} M_{\lambda-1}$. Thus, for consistency we must have $\mu_{\lambda-1} = 1$ and $\mu_\lambda = 1$. These are the normalization conditions that replace $\mu_0 = 1$ and $\mu_1 = 1$ that are implicit in the volume-conserving Friedlander ansatz.

Substitution of (3.1) into the moment equation (2.3) gives

$$\mu_k M_{\lambda-1}^{\alpha-1}(t)\frac{\mathrm{d}M_{\lambda-1}(t)}{\mathrm{d}t} = \tilde{c}_\alpha A_k \mu_{k+\alpha} M_\lambda^\alpha. \tag{3.4}$$

Taking $k = \lambda - 1$ in (3.4) and using $\mu_{\lambda-1} = 1$ we obtain

$$M_{\lambda-1}^{\alpha-1}(t)\frac{\mathrm{d}M_{\lambda-1}(t)}{\mathrm{d}t} = \tilde{c}_\alpha A_{\lambda-1} \mu_{\lambda-1+\alpha} M_\lambda^\alpha. \tag{3.5}$$

One can check the units of the moment equations by recalling that $\dim(M_k) = x^{k-1}$, $\dim(\tilde{c}_\alpha) = x^{-\alpha}t^{-1}$ and all other quantities are dimensionless.

The integral of (3.5) is

$$M_{\lambda-1}(t) = M_{\lambda-1}(0)(1 + \beta^\alpha \varsigma)^{1/\alpha} \tag{3.6}$$

where

$$\beta^\alpha = \alpha \mu_{\alpha+\lambda-1} A_{\lambda-1} \tag{3.7}$$

is a significant dimensionless parameter and $\varsigma = \tilde{c}_\alpha v_\lambda^\alpha t$ is a dimensionless time. The parameter $v_\lambda = M_\lambda / M_{\lambda-1}(0)$ is the initial mean value of x^λ and is akin to the initial mean particle volume $v_1 = V(0)/N(0)$ and reduces to it when $\lambda \to 1$. Substitution of (3.6) into (3.2) gives the solution for all the moments as

$$M_k(t) = M_k(0)(1 + \beta^\alpha \varsigma)^{-(k-\lambda)/\alpha} \tag{3.8}$$

where

$$M_k(0) = \mu_k M_{\lambda-1}(0) v_\lambda^{k+1-\lambda} \tag{3.9}$$

are the initial values of the moments of the similarity solution. In particular, we see from (3.8) that the number and volume are given by

$$N(t) = N(0)(1 + \beta^\alpha \varsigma)^{\lambda/\alpha} \qquad V(t) = V(0)(1 + \beta^\alpha \varsigma)^{-(1-\lambda)/\alpha}. \tag{3.10}$$

By inspection of (3.10) it is evident that a decrease in volume ($\lambda < 1$) slows the growth of the particle number and *vice versa* if the volume would be increasing. As is evident from (3.8), the exponent for ratios of moments is independent of λ. For instance for the instantaneous mean particle volume $v_1(t)$ and the $v_\lambda(t)$ mean value we have

$$v_1(t) = \frac{V(t)}{N(t)} = v_1(1 + \beta^\alpha \varsigma)^{-1/\alpha} \qquad v_\lambda(t) = \frac{M_\lambda}{M_{\lambda-1}(t)} = v_\lambda(1 + \beta^\alpha \varsigma)^{-1/\alpha}. \tag{3.11}$$

To emphasize the generality of the moment solution (3.8) we point out that in the derivation we did not need to assume a specific function form for the daughter distribution. Nor did we need to know specifically the reduced distribution $\phi(z)$, only that it exists and its moments (3.3) are finite for $k \geqslant \lambda - 1$. However, the allowed numerical ranges for λ and the other fragmentation parameters depend on the daughter distribution and will be determined in the construction of the solution for the reduced distribution.

3.1. The reduced moment equation and the reduced moments for the polynomial daughter distribution

By comparing (3.4) and (3.5) we obtain the fundamental recursion relation

$$\mu_{k+\alpha} = \frac{\beta^\alpha}{\alpha A_k}\mu_k \tag{3.12}$$

where we used $\beta^\alpha = \alpha\mu_{\lambda-1+\alpha}A_{\lambda-1}$. Starting with $k = \lambda - 1$ and iterating (3.12) in steps of α we obtain the solution of (3.12) which is

$$\mu_{\lambda-1+n\alpha} = \frac{\beta^{n\alpha}}{\alpha^n A_{\lambda-1}A_{\lambda-1+\alpha}\ldots A_{\lambda-1+(n-1)\alpha}}. \tag{3.13}$$

As in the analysis in [15] for volume-conserving fragmentation we assume the polynomial form of the daughter-fragment distribution

$$b(r) = r^\gamma(b_0 + b_1 r + \cdots + b_p r^p) \tag{3.14}$$

where p is a positive integer and γ and the coefficients $b_0, b_1, \ldots, b_p$ are real. Substitution into (2.4) and integration gives

$$\begin{aligned}(k-\lambda)A_k &= 1 - \int_0^1 r^k b(r)\,\mathrm{d}r \\ &= 1 - \frac{b_0}{k+\gamma+1} - \frac{b_1}{k+\gamma+2} - \cdots - \frac{b_p}{k+\gamma+p+1}.\end{aligned} \tag{3.15}$$

There are $p+1$ zeros of $(k-\lambda)A_k$. One of them, λ, has been assumed to be positive and we now suppose that the other p zeros are real and negative. These conditions can be satisfied by proper choice of the b coefficients. Thus, we may write A_k in the factored form

$$A_k = \frac{(k+\lambda_1)(k+\lambda_2)\ldots(k+\lambda_p)}{(k+\gamma+1)(k+\gamma+2)\ldots(k+\gamma+p+1)} \tag{3.16}$$

where we suppose that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are real, positive numbers and thus $-\lambda_1, -\lambda_2, \ldots, -\lambda_p$ are the zeros of A_k. From (3.15) we see that the positive zero and the negative zeros are solutions of

$$\frac{b_0}{k+\gamma+1} - \frac{b_1}{k+\gamma+2} - \cdots - \frac{b_p}{k+\gamma+1+p} = 1. \tag{3.17}$$

We show in appendix B that for A_k given by (3.16) the solution (3.13) for the μ-moments is given by

$$\begin{aligned}\mu_{\lambda-1+k} = D_p(\alpha,\gamma,\lambda)\beta^k &[\Gamma((k+\gamma+\lambda)/\alpha)\Gamma((k+\gamma+\lambda+1)/\alpha)\ldots \\ &\ldots\Gamma((k+\gamma+\lambda+p)/\alpha)] \\ &\times[\Gamma((k+\lambda_1+\lambda-1)/\alpha)\Gamma((k+\lambda_2+\lambda-1)/\alpha)\ldots \\ &\ldots\Gamma((k+\lambda_p+\lambda-1)/\alpha)]^{-1}\end{aligned} \tag{3.18}$$

where Γ is the gamma function and

$$D_p(\alpha,\gamma,\lambda) = \frac{\Gamma((\lambda_1+\lambda-1)/\alpha)\Gamma((\lambda_2+\lambda-1)/\alpha)\ldots\Gamma((\lambda_p+\lambda-1)/\alpha)}{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\gamma+\lambda+1)/\alpha)\ldots\Gamma((\gamma+\lambda+p)/\alpha)} \tag{3.19}$$

is independent of k. For brevity we will refer to $\lambda_1, \lambda_2, \ldots, \lambda_p$ as zeros of A_k even though $-\lambda_1, -\lambda_2, \ldots, -\lambda_p$ are the values for which $A_k = 0$. Recalling that we have normalized with $\mu_\lambda = 1$, it follows from (3.18) and (3.19) that

$$\beta = \frac{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\lambda+\lambda_1)/\alpha)\Gamma((\lambda+\lambda_2)/\alpha)\ldots((\lambda+\lambda_p)/\alpha)}{\Gamma((\gamma+\lambda+1+p)/\alpha)\Gamma((\lambda_1+\lambda-1)/\alpha)\Gamma((\lambda_2+\lambda-1)/\alpha)\ldots\Gamma((\lambda_p+\lambda-1)/\alpha)}. \tag{3.20}$$

We see that for $\gamma + \lambda = 0$, because of the singularity of $\Gamma((\gamma+\lambda)/\alpha)$, that $\beta = \infty$. To avoid this singularity and to insure that $\beta < \infty$ we impose the constraint $\gamma + \lambda > 0$. The inverse Mellin transform of (3.18) gives the solution for the reduced distribution ϕ as we now show.

4. The solution for the reduced distribution $\phi(z)$

The inverse Mellin transform is given by

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} z^{-(k+1)} \mu_k \, \mathrm{d}k \tag{4.1}$$

where Br is the Bromwich path in the complex k-plane (from $k = -\mathrm{i}\infty$ to $k = +\mathrm{i}\infty$ to the right of the singularities of μ_k) and μ_k is given by (3.18) with k continued to the Bromwich path.

Substitution of (3.18) into the inversion integral (4.1) and making a change of the variable of integration gives

$$\begin{aligned}\overline{\phi}(\eta) = D_p(\alpha, \gamma, \lambda) \frac{\alpha}{\beta^{\lambda}} \eta^{\gamma/\alpha} \\ \times \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \eta^{-k} [\Gamma(k)\Gamma(k + 1/\alpha) \ldots \Gamma(k + p/\alpha)] \\ \times [\Gamma(k + (\lambda_1 - \gamma - 1)/\alpha)\Gamma(k + (\lambda_2 - \gamma - 1)/\alpha) \ldots \\ \ldots (k + (\lambda_p - \gamma - 1)/\alpha)]^{-1} \, \mathrm{d}k\end{aligned} \tag{4.2}$$

where $\sigma > 0$, $\eta = z^{\alpha}/\beta^{\alpha}$ is the scaled similarity variable and $\overline{\phi}(\eta) = \phi(z)$. Fortunately, the inversion integral in (4.2) is a particular case of a known special function called a Meijer G-function. An analysis of the general G-function and its properties are given by Luke [18]. There one finds the definition of our particular G-function as

$$\begin{aligned}G^{p+1,0}_{p,p+1}\left(\frac{\lambda_p - \gamma - 1}{\alpha}; 0, \frac{p}{\alpha}; \eta\right) \\ = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \eta^{-k} [\Gamma(k)\Gamma(k + 1/\alpha) \ldots \Gamma(k + p/\alpha)] \\ \times [\Gamma(k + (\lambda_1 - \gamma - 1)/\alpha)\Gamma(k + (\lambda_2 - \gamma - 1)/\alpha) \ldots \\ \ldots (k + (\lambda_p - \gamma - 1)/\alpha)]^{-1} \, \mathrm{d}k\end{aligned} \tag{4.3}$$

where we have used the short notation

$$\begin{aligned}G^{p+1,0}_{p,p+1}\left(\frac{\lambda_p - \gamma - 1}{\alpha}; 0, \frac{p}{\alpha}; \eta\right) \\ = G^{p+1,0}_{p,p+1}\left(\frac{\lambda_1 - \gamma - 1}{\alpha}, \frac{\lambda_2 - \gamma - 1}{\alpha}, \ldots, \frac{\lambda_p - \gamma - 1}{\alpha}; 0, \frac{1}{\alpha}, \frac{2}{\alpha}, \ldots \frac{p}{\alpha}; \eta\right).\end{aligned}$$

The lower-inner index p is the number of gamma-function factors in the denominator and the first p arguments of the G-function show the arguments of the denominator factors. The upper-inner index $p+1$ is the number of gamma-function factors in the numerator and the next $p+1$ arguments of G, including the zero, show the positions of the poles of the numerator factors. The upper-outer index being zero means that there are no more gamma function factors in the numerator and the difference of the upper-inner and lower-outer indices being zero means that there are no other gamma-function factors in the denominator. For $p = 0$ the G-function (4.3) is the exponential function, for $p = 1$ it is a sum of two ${}_1F_1$ confluent hypergeometric series and for $p > 1$ it is a linear combination of ${}_pF_p$ generalized confluent hypergeometric series, where

$${}_pF_p(a_1, a_2, \ldots, a_p; c_1, , \ldots, c_p; \eta) = \sum_{n=0}^{n=\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(c_1)_n (c_2)_n \ldots (c_p)_n} \frac{\eta^n}{n!}$$

and $(a)_n = a(a+1)\ldots(a+n-1)$ is the Pochhammer factorial, which is related to the gamma function by $\Gamma(a+n) = \Gamma(a)(a)_n$.

With (4.3), we write (4.2) as

$$\overline{\phi}(\eta) = D_p(\alpha,\gamma,\lambda)\frac{\alpha}{\beta^{\lambda}}\eta^{\gamma/\alpha}G^{p+1,0}_{p,p+1}\left(\frac{\lambda_p-\gamma-1}{\alpha};0,\frac{p}{\alpha};\eta\right). \tag{4.4}$$

The solution depends on λ in the coefficient of the G-function and depends on β in the scaled similarity variable in the argument of the G-function which depends on λ. The solution (4.4) is the same as the solution for volume-conserving fragmentation given in [15] except that it contains the parameter λ, which now need not be unity and the parameters $\lambda_1, \lambda_2, \ldots, \lambda_p$ (the zeros of A_k) in the argument of G are shifted from what they would be for volume-conserving fragmentation. The solution (4.4) reduces to the solution for volume-conserving fragmentation if everywhere in the solution and in the determination of the zeros by (3.17) one takes $\lambda = 1$.

There are other representations of the G-function besides the one given by the inversion integral (4.3). Two alternative representations have been derived in [15] which, for convenience, we show again in appendix C. One of these gives $G^{p+1,0}_{p,p+1}$ as a linear combination of ${}_pF_p$ generalized hypergeometric functions from which one obtains the small-η limit. The other is a real multiple integral representation that shows the constraints on the zeros of A_k and is used to derive the large-η asymptotic limit of the solution. We will show the details of both of these representations for the $p = 1$ case.

4.1. Constraints on the parameters

One sees by inspection of the integral representation given in appendix C by equation (C.4) that a necessary condition for the existence of the integrals is

$$\lambda_1 > \gamma+2, \lambda_2 > \gamma+3, \ldots, \lambda_p > \gamma+1+p. \tag{4.5}$$

It follows from (3.15) and (3.16) with $k = 0$ that the fragment number is given by

$$\tilde{N} = \int_0^1 b(r)\,\mathrm{d}r = 1 + \lambda A_0 = 1 + \frac{\lambda\lambda_1\lambda_2\ldots\lambda_p}{(\gamma+1)(\gamma+2)\ldots(\gamma+1+p)}. \tag{4.6}$$

If one wants two or finitely more fragments per fragmentation then we have the further constraint

$$\frac{\lambda\lambda_1\lambda_2\ldots\lambda_p}{(\gamma+1)(\gamma+2)\ldots(\gamma+1+p)} \geqslant 1 \qquad \gamma > -1. \tag{4.7}$$

As we noted above, for finite β it is necessary that

$$\gamma + \lambda > 0. \tag{4.8}$$

We impose the constraints (4.5), (4.7) and (4.8) on the fragmentation parameters.

4.2. The reduced equation for ϕ

One of the curious features of the similarity solution is that by applying the Mellin transformation and its inversion we have been able to construct the solution without ever considering the ordinary differential equation satisfied by the reduced distribution $\phi(z)$. As noted in the introduction, the equation satisfied by $\phi(z)$ is obtained by substituting the similarity form (3.1) into the fragmentation equation (1.1) and separating the variables z

and t. Carrying out this substitution one confirms that the similarity variable and the time separate and yields the *reduced equation*

$$z\frac{\mathrm{d}\phi(z)}{\mathrm{d}z}+(1+\lambda)\phi(z)=\frac{\alpha}{\beta^{\alpha}}\left[-z^{\alpha}\phi(z)+\int_{z}^{\infty}b\left(\frac{z}{w}\right)w^{\alpha-1}\phi(w)\,\mathrm{d}w\right] \tag{4.9}$$

where β^{α} now appears as a separation parameter. When the proper boundary conditions are enforced the solution of (4.9) should be (4.4). Thus, another approach to constructing the solution $\phi(z)$ (or equivalently $\overline{\phi}(\eta)$) would be to try to solve (4.9) as a linear combination of ${}_pF_p$ series. For consistency one should obtain the same solution that was obtained by the Mellin transformation. However, we have not tried to construct the solution in this way, rather, as a simple check for consistency, we have taken moments of (4.9) and find again the recursion formula (3.12) for μ_k, as we should. In appendix D we show that the recursion equation (3.12) for μ_k is obtained by taking moments of (4.9).

5. The small- and large-η behaviour for $\overline{\phi}(\eta)$ and the long-time limit for $n(x,t)$

5.1. The small-η limit for $\overline{\phi}(\eta)$

The power series representation of $G^{p+1,0}_{p,p+1}(\eta)$ at $\eta=0$ for a general polynomial $b(r)$ is given in appendix C where we show the leading terms of the series. From (C.1) and (4.4) we obtain the limit

$$\lim_{n\to 0}\overline{\phi}(\eta)=C_0\eta^{\gamma/\alpha} \tag{5.1}$$

where C_0 is a constant that is determined by (3.19), (4.4) and the leading term in (C.1). Thus we have the general result that the small-η power-law behaviour is determined by r^{γ} (the leading term in the daughter-fragment distribution) and does not depend explicitly on the coefficients of the polynomial factor in $b(r)$. However, the values of γ are constrained by (4.5), (4.7) and (4.8) and in this sense depend on the other fragmentation parameters in the polynomial factor. We will see that by allowing a larger range for γ that the polynomial factor can have a significant effect on the small-η behaviour.

5.2. The large-η asymptotic limit for $\phi(\eta)$

An integral representation for the $G^{p+1,0}_{p,p+1}(\eta)$ was derived in [15] for $\lambda=1$ and with only a change in notation is given for $\lambda\neq 1$ by equation (C.4) in appendix C. Following the same steps given in [15], the limit for large-η obtained from the integral representation is

$$\lim_{\eta\to\infty}\overline{\phi}(\eta)\sim C_{\infty}\frac{\alpha}{\beta^{\lambda}}\eta^{(\gamma+\Lambda)/\alpha}\exp(-\eta) \tag{5.2}$$

where C_{∞} is a constant,

$$\Lambda=-\sum_{j=1}^{p}(\lambda_j-\gamma-1-j) \tag{5.3}$$

and the lower bounds on λ_j, $j=1,2,\ldots,p$, are given by (4.5). It follows from (5.3) and (4.5) that $\Lambda<0$ and hence the limit (5.2) is bounded from above by constant $\times\,\eta^{\gamma/\alpha}\exp(-\eta)$.

According to Luke [18, section 5.10, p 189], except for $p=0$, for which $G^{1,0}_{0,1}$ is the exponential function, the large-η expansion of the G-function is an asymptotic expansion, as we have indicated by the notation. Thus, the limit (5.2) is the limit of an asymptotically converging series in η^{-1}. In appendix C for the special case $p = 1$ we confirm the asymptotic nature of the expansion. There we show that

$$G^{2,0}_{1,2}\left(\frac{\lambda_1-\gamma-1}{\alpha};0,\frac{1}{\alpha};\eta\right)$$
$$\sim \eta^{-(\lambda_1-\gamma-2)/\alpha}\exp(-\eta)\,{}_2F_0\left(\frac{\lambda_1-\gamma-2}{\alpha},\frac{\lambda_1-\gamma-1}{\alpha};;-\eta^{-1}\right) \tag{5.4}$$

where

$${}_2F_0\left(\frac{\lambda_1-\gamma-2}{\alpha},\frac{\lambda_1-\gamma-1}{\alpha};;-\eta^{-1}\right)=\sum_{n=0}^{n=\infty}\frac{(\lambda_1-\gamma-2)_n(\lambda_1-\gamma-1)_n}{n!}(-\eta)^{-\eta}. \tag{5.5}$$

Because of the two Pochhammer factorials in the numerator it is evident that the series diverges for all η, but it converges asymptotically.

5.3. The long-time limit of the similarity solution for all η

We recall that the full similarity solution is given by (3.1). Substitution of the moment solution (3.6) into (3.1) gives

$$n(x,t)=\frac{M_\lambda}{v_\lambda^\lambda}(1+\beta^\alpha\varsigma)^{(\lambda+1)/\alpha}\overline{\phi}(\eta) \tag{5.6}$$

where

$$\eta=\frac{z^\alpha}{\beta^\alpha}=\frac{1}{\beta^\alpha}\left(\frac{M_{\lambda-1}(t)}{M_\lambda}\right)^\alpha x^\alpha=\frac{1}{\beta^\alpha v_\lambda^\alpha}(1+\beta^\alpha\varsigma)x^\alpha. \tag{5.7}$$

Thus, except for the dependence of v_λ and β on λ, the similarity variables for fragmentation with constant volume and fragmentation with volume change are the same functions of ς. For $\lambda\to 1$, (5.7) reduces to the Friedlander similarity variable.

In the limit $\varsigma=\tilde{c}_\alpha v_\lambda^\alpha t\to\infty$ we have

$$\lim_{\varsigma\to\infty}\eta=\frac{\varsigma x^\alpha}{v_\lambda^\alpha}=\tilde{c}_\alpha x^\alpha t$$
$$\lim_{\varsigma\to\infty}\frac{M_{\lambda-1}^{\lambda+1}(t)}{M_\lambda^\lambda}=\frac{M_\lambda}{v_\lambda^{\lambda+1}}\lim_{\varsigma\to\infty}(1+\beta^\alpha\varsigma)^{(1+\lambda)/\alpha}=\frac{M_\lambda}{v_\lambda^{\lambda+1}}\beta^{(1+\lambda)}\varsigma^{(1+\lambda)/\alpha} \tag{5.8}$$

where we see that β and v_λ cancel out of the scaled similarity variable. To denote the limit of the similarity variable we use the notation $\eta_\infty=\lim_{\varsigma\to\infty}\eta=\tilde{c}_\alpha x^\alpha t$, which is just ς with the initial average 'volume' v_λ replaced by the arbitrary particle volume x. Thus, with (5.6) and (5.8) we obtain the limit

$$\lim_{\varsigma\to\infty}n(x,t)=C\varsigma^{(\lambda+1)/\alpha}\overline{\phi}(\eta_\infty)\qquad\varsigma=\tilde{c}_\alpha v_\lambda^\alpha t\qquad\eta_\infty=\tilde{c}_\alpha x^\alpha t \tag{5.9}$$

where C is a constant. Thus, if $\beta^\alpha\varsigma\gg 1$ and $(\alpha\beta^\alpha\varsigma)/(\lambda+1)\gg 1$ then (5.6) may be represented by (5.9). Usually the long-time behaviour is emphasized and then the parameter β does not appear in the analysis. However, from (5.6) and (5.7) we see that β is significant for setting the time scale of the fragmentation process. For example, because of the pole at $\gamma+\lambda=0$ in the gamma function in the numerator in (3.20), if $\gamma+\lambda\to 0$ then $\beta\to\infty$ and consequently in a very short time the distribution approaches the long-time limit. The parameter v_λ is usually not shown; however, we show it here because it affects the fragmentation time scale.

The mean value v_λ appearing in solution (5.6) is an initial condition that is independent of the fragmentation physics and may be chosen arbitrarily. Thus, the similarity solution is actually a one-parameter family of solutions labelled with the parameter v_λ. On account of the linearity of the equation one can form additional solutions by the superposition of similarity solutions with different values of v_λ. However, for long times, as we have seen in (5.9), v_λ cancels out of the solution and thus, up to a constant multiplicative factor, solutions with different v_λ tend to the same limit. Therefore, different linear combinations of similarity solutions tend to the same limit. This is an example of Fillipov's theorem [1], which states that solutions of the fragmentation equation with different initial conditions tend to the same limit distribution.

5.4. The long-time, small-η limit and the long-time, large-η limit

When considered as a function of η inspection of the similarity form shows that one may think of $(M_\lambda/v_\lambda^{\lambda+1})(1+\beta^\alpha\varsigma)^{(\lambda+1)/\alpha}$ as the amplitude of the distribution. The amplitude of the distribution changes in time but the shape of the distribution as a function of η does not change and hence the name similarity or self-similar. One can also consider the distribution as a function of x, or better as a function of $(x/v_\lambda)^\alpha$, with ς a parameter. Then, as a function of $(x/v_\lambda)^\alpha$ with increasing ς the shape of the distribution changes by becoming narrower. As ς increases the amplitude of $n(x,t)$ increases (because of the increase in the number of particles) and the distribution ϕ as a function of $(x/v_\lambda)^\alpha$ becomes narrower in such a way that the moment M_λ remains constant. As the distribution becomes narrower there is still a front and a tail of the distribution where the same definitions of front ($\eta \ll 1$) and tail ($\eta \gg 1$) continue to apply. (If one wants to be more precise then one can inspect the leading terms in the small η expansion and the large η asymptotic expansion.) In the limit $\varsigma \to \infty$ the amplitude of the distribution tends to infinity with M_λ constant. Thus we may consider the function $s_t(x) = x^\lambda n(x,t)$ as a sequence of functions of x with t (or ς) a parameter. As t increases the functions of x become narrower and increase in amplitude while the integral over x remains constant and for $t \to \infty$ the sequence $s_t(x)$ is localized in the positive neighbourhood of $x = 0$. This behaviour is like a δ-sequence [19], where the limit as $t \to \infty$ is the Dirac delta-function $\delta(x)$. From a physical point of view this is a natural limit for a system of particles that continues to fragment with no smallest particle cut-off in the fragmentation process. For $t \to \infty$, the moments M_k, $k > \lambda$ vanish, the moments M_k, $k < \lambda$ diverge and for the special value $k = \lambda$, the moment remains constant, which is the behaviour of the moments of a Dirac delta-function.

Another simple feature of the fragmentation kinetics is shown by (3.11), where for $\beta^\alpha\varsigma/\alpha \gg 1$ we see that the instantaneous mean volume of the distribution approaches zero according to the power law $v_1(t) = v_1\beta\varsigma^{-1/\alpha}$, where v_1 is the initial mean volume. We see that this behaviour is independent of all fragmentation parameters except the degree of homogeneity α, i.e. independent of parameters that characterize the daughter-fragment distribution. This behaviour is well known and has been deduced from the scale invariance of volume-conserving fragmentation.

If there is interest in the dependence of the distribution separately on x and ς for long times, one can make this separation in the limits (5.1), (5.2) and (5.6). Then for large ς and small η_∞ we have

$$\lim_{\varsigma\to\infty,\eta_\infty\to 0} n(x,t) = C_0'\varsigma^{(\lambda+1+\gamma)/\alpha}(x/v_\lambda)^\gamma \tag{5.10}$$

where C_0' is a constant. For large ς and large η_∞ we have

$$\lim_{\varsigma\to\infty,\eta_\infty\to\infty} n(x,t) \sim C_\infty'\varsigma^{(\lambda+1+\gamma+\Lambda)/\alpha}\left(\frac{x}{v_\lambda}\right)^{\gamma+\Lambda}\exp\left(-\frac{\varsigma x^\alpha}{v_\lambda^\alpha}\right) \tag{5.11}$$

where C'_∞ is a constant.

We now give examples where we will see in a simple setting how the solutions depend on the fragmentation parameters γ, λ and λ_1 and we will show the allowed numerical range of γ, λ and λ_1.

6. Examples for $p = 0$ and $p = 1$

The simplest cases are the power law ($p = 0$) and the linear ($p = 1$) daughter distributions.

6.1. Power law, $p = 0$, $b(r) = b_0 r^\gamma$

According to (4.3) with $p = 0$ the G-function is given by

$$G^{1,0}_{0,1}(0; 0; \eta) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Br}} \eta^{-k} \Gamma(k)\,\mathrm{d}k = \exp(-\eta). \tag{6.1}$$

With (4.4) and (6.1) we have the reduced distribution

$$\overline{\phi}(\eta) = \frac{\alpha}{\Gamma((\gamma+\lambda)/\alpha)\beta^\lambda} \eta^{\gamma/\alpha} \exp(-\eta) \tag{6.2}$$

and with (3.1) the similarity solution is

$$n(x,t) = \frac{M_\lambda}{v_\lambda^{\lambda+1}} (1 + \beta^\alpha \varsigma)^{(\lambda+1)/\alpha} \overline{\phi}(\eta). \tag{6.3}$$

For $p = 0$ we have

$$b_0 = \lambda + \gamma + 1 \qquad \tilde{N} = 1 + \frac{\lambda}{\gamma+1} \qquad \beta = \frac{\Gamma((\gamma+\lambda)/\alpha)}{\Gamma((\gamma+\lambda+1)/\alpha)} \tag{6.4}$$

where the parameters satisfy the constraints

$$\gamma > -1 \qquad \gamma + \lambda > 0 \qquad \frac{\lambda}{\gamma+\lambda} \geqslant 1. \tag{6.5}$$

For $(\lambda + 1)\beta^\alpha \varsigma/\alpha \gg 1$, $\beta^\alpha \varsigma \gg 1$ we obtain the long-time limit from (6.2) and (5.8). If $\lambda = 1$ then $-1 < \gamma \leqslant 0$, $\tilde{N}$ and β are finite, there are two or more fragments per fragmentation and from (6.2) and (6.3) we recover the volume-conserving solution given by Peterson [3].

Equation (3.8) gives the solution for the moments for all p. By inspection of (3.8) we see that the characteristic real time for the change in the moments is

$$t_k^* = \frac{\alpha}{(k-\lambda)\beta^\alpha \tilde{c}_\alpha v_\lambda^\alpha}$$

where the moments with k close to λ of course have large time constants. Or, if $\gamma \to -\lambda$ the parameter β becomes large, the time constant becomes small and there is a very rapid change of the moments except for k very close to λ. As we see the characteristic time is determined by the allowed numerical values of α, γ, λ, the rate constant $\tilde{c}_\alpha$ and the initial mean moment v_λ. The mean moment v_λ is an initial condition but the parameters α, γ, λ and $\tilde{c}_\alpha$ can only be obtained by an analysis of the physics of the fragmentation process, or by fitting to measurements on the evolution of the distribution or measurements of the moments of the distribution.

6.2. Linear daughter distribution, $p = 1$, $b(r) = r^{\gamma}(b_0 + b_1 r)$

For the linear daughter distribution one may take γ, b_0, b_1 as independent parameters and thereby determine the parameters $\gamma, \lambda, \lambda_1$. Or, one may take $\gamma, \lambda, \lambda_1$ as independent which is our point of view. Then the b coefficients are determined by $\gamma, \lambda, \lambda_1$. To see this we recall that the positive zero $k = \lambda$ and the negative zero $k = -\lambda_1$ are solutions of (3.17), which for the linear daughter distribution is

$$\frac{1}{(\lambda+\gamma+1)}b_0 + \frac{1}{(\lambda+\gamma+2)}b_1 = 1 \qquad \frac{1}{(-\lambda_1+\gamma+1)}b_0 + \frac{1}{(-\lambda+\gamma+2)}b_1 = 1. \tag{6.6}$$

The unique solution is

$$b_0 = -(\lambda+\gamma+1)(-\lambda_1+\gamma+1) \qquad b_1 = (\lambda+\gamma+2)(-\lambda_1+\gamma+2). \tag{6.7}$$

Alternatively, if one knew both the b coefficients and the parameter γ from a physical model of the fragmentation process then one could solve (6.6) for λ and λ_1.

Continuing with the solution, from (4.4) for $p = 1$ we have

$$\overline{\phi}(\eta) = \frac{\Gamma((\lambda_1+\lambda-1)/\alpha)}{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\gamma+\lambda+1)/\alpha)}\frac{\alpha}{\beta^{\lambda}}\eta^{\gamma/\alpha}G^{2,0}_{1,2}\left(\frac{\lambda_1-\gamma-1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right). \tag{6.8}$$

With the expansion of $G^{2,0}_{1,0}(\eta)$ for small η in powers of η given in appendix C, the representation of $\overline{\phi}$ is

$$\overline{\phi}(\eta) = \frac{\Gamma((\lambda_1+\lambda-1)/\alpha)}{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\gamma+\lambda+1)/\alpha)}\frac{\alpha}{\beta^{\lambda}}\eta^{\gamma/\alpha}\exp(-\eta)\psi\left(\frac{\lambda_1-\gamma-2}{\alpha}; 1-\frac{1}{\alpha}; \eta\right). \tag{6.9}$$

where

$$\begin{aligned}\exp(-\eta)\psi&\left(\frac{\lambda_1-\gamma-2}{\alpha}; 1-\frac{1}{\alpha}; \eta\right)\\ &= \frac{\Gamma(1/\alpha)}{\Gamma((\lambda_1-\gamma-1)/\alpha)}{}_1F_1\left(1-\frac{\lambda_1-\gamma-1}{\alpha}; 1-\frac{1}{\alpha}; -\eta\right)\\ &\quad+\frac{\Gamma(-1/\alpha)\eta^{1/\alpha}}{\Gamma((\lambda_1-\gamma-2)/\alpha)}{}_1F_1\left(1-\frac{\lambda_1-\gamma-2}{\alpha}; 1+\frac{1}{\alpha}; -\eta\right).\end{aligned}$$

The function ψ, sometimes called the ψ-function, is a well known special function [18, 19]. From (6.9) we see that the $\eta \to 0$ limit is

$$\lim_{n\to 0}\overline{\phi}(\eta) = \frac{\Gamma((\lambda_1+\lambda-1)/\alpha)}{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\gamma+\lambda+1)/\alpha)}\frac{\Gamma(1/\alpha)}{\Gamma((\lambda_1-\gamma-1)/\alpha)}\frac{\alpha}{\beta^{\lambda}}\eta^{\gamma/\alpha}. \tag{6.10}$$

With the asymptotic expansion of $G^{2,0}_{1,0}(\eta)$ in powers of η^{-1} given in appendix C we have

$$\begin{aligned}\overline{\phi}(\eta) \sim{}& \frac{\Gamma((\lambda_1+\lambda-1)/\alpha)}{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\gamma+\lambda+1)/\alpha)}\frac{\alpha}{\beta^{\lambda}}\eta^{-(\lambda_1-2\gamma-2)/\alpha}\\ &\times\exp(-\eta){}_2F_0\left(\frac{\lambda_1-\gamma-2}{\alpha}, \frac{\lambda_1-\gamma-1}{\alpha}; ; -\eta^{-1}\right)\end{aligned} \tag{6.11}$$

which is the asymptotic expansion for large η.

The similarity solution is given by

$$n(x,t) = \frac{M_{\lambda}}{v_{\lambda}^{\lambda+1}}(1+\beta^{\alpha}\varsigma)^{(\lambda+1)/\alpha}\overline{\phi}(\eta)$$

where $\overline{\phi}$ is given by (6.9) and from (3.20) with $p = 1$

$$\beta = \frac{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\lambda+\lambda_1)/\alpha)}{\Gamma((\gamma+\lambda+1+p)/\alpha)\Gamma((\lambda_1+\lambda-1)/\alpha)}.$$

According to (4.5), (4.7) and (4.8) the parameter constraints are $\gamma > -1$ and

$$\lambda_1 > \gamma + 2 \qquad \frac{\lambda\lambda_1}{(\gamma+1)(\gamma+2)} \geqslant 1 \qquad \gamma + \lambda > 0.$$

Ziff and McGrady [4] were the first to derive a solution of the fragmentation equation for the linear daughter distribution. As they noted, the linear $b(r)$ allows a large positive γ, whereas for the power law we have $-1 < \gamma \leqslant 0$. The larger range of γ for the solution for the linear $b(r)$ persists for the polynomial $b(r)$. This behaviour should be of interest because it allows distributions that vanish in the small-η limit (6.10), which on physical grounds would seem to be more reasonable behaviour than the constant or infinite limits of $n(x,t)$ allowed by the power-law solution.

To show a specific example, we take the allowed values $\alpha = 3$, $\gamma = 1$, $\lambda = 1$, $\lambda_1 = 6$. This gives binary fragmentation with

$$\tilde{N} = 1 + \frac{\lambda\lambda_1}{(\gamma+1)(\gamma+2)} = 2$$

$$\beta = \frac{\Gamma((\gamma+\lambda)/\alpha)\Gamma((\lambda+\lambda_1)/\alpha)}{\Gamma((\gamma+\lambda+2)/\alpha)\Gamma((\lambda_1+\lambda-1)/\alpha)} = \frac{4}{3}\Gamma\left(\frac{2}{3}\right).$$

Then (6.9) reduces to a solution first given by Ziff and McGrady [4] and since $\gamma = 1$ their solution has the limiting behaviour $\lim_{\eta\to 0} n(x,t) = 0$.

7. Summary and discussion

We have derived a generalization of the Friedlander similarity form of distribution (given by equation (3.1)) for fragmentation with volume change and we have derived a solution (given by equation (3.8)) for the time dependence of the moments of the similarity distribution for a general daughter distribution. By inversion of the Mellin transformation of the fragmentation we have obtained an exact solution of the equation for a polynomial distribution of daughter fragments of arbitrarily large polynomial degree p. The solution is a generalization of known solutions.

The limits for small and large values of the similarity variable η and the long-time limit in the time variable ς follow rigorously from the exact solution and are given by (5.1), (5.2) and (5.9). The limits for the similarity variable hold for all times $0 \leqslant \varsigma \to \infty$ and the long-time limit holds for all values of the similarity variable. A novel feature of the analysis is that by constructing the solution for all times rather than looking only at the long-time limit we have been able to identify the time constants for changes in the distribution and the moments, which are sensitive to the values of the fragmentation parameters.

A single parameter λ determines volume change. If $\lambda = 1$ volume is conserved, if $\lambda \neq 1$ volume is not conserved and instead the moment M_λ is conserved. The effect of volume change on the distribution and the moments of the distribution is expressed by the numerical value of λ that appears explicitly in the solution.

The details of the similarity solution may be seen from the series solutions, the small- and large η-limits and the $\varsigma \to \infty$ limit that have been given. However, a qualitative picture of the behaviour of the distribution in time can be seen just from the form of the similarity solution. As discussed in section 5, by regarding $s_t(x) = x^\lambda n(x,t)$ as a time sequence of functions of the particle volume x one sees that as time increases the distribution becomes narrower and the amplitude increases in such a way that the moment distribution becomes narrower and the moment M_λ is constant. Then, $\lim_{t\to\infty} x^\lambda n(x,t)$ behaves like a δ-sequence where the limit is the Dirac delta-function.

Generalizing the daughter distribution to a polynomial form that allows a change in volume (as well as no change) has introduced new parameters and new allowed numerical ranges of the parameters into the theory thereby increasing the flexibility of the theory to match conditions of real fragmentation phenomena.

Acknowledgment

It is a pleasure to thank Dr H A Weldon for a number of helpful discussions on the analysis.

Appendix A. The Friedlander similarity form for fragmentation with volume change

We consider the scale transformations of time and particle volume given by

$$t = r^{-\alpha}(t^* + t_0) \qquad x = rx^*. \tag{A.1}$$

Substitution into the fragmentation equation (1.1) gives, after relabelling the variables, the transformed equation

$$\frac{\partial}{\partial t} n_r(x,t) = -\tilde{c}_\alpha x^\alpha n_r(x,t) + \tilde{c}_\alpha \int_x^\infty y^{\alpha-1} b(x/y) n_r(y,t)\,\mathrm{d}y \tag{A.2}$$

where the transformed distribution is given by

$$n_r(x,t) = r^{1+\lambda} n[rx, r^{-\alpha}(t+t_0)]. \tag{A.3}$$

Equation (A.2) is identical to (1.1), which is to say that (1.1) is invariant under the scale transformation. The factor in the transformed distribution is arbitrary but has been chosen so that the λ-moment of the transformed distribution, $M_{r,\lambda} = \int_0^\infty x^\lambda n_r(x,t)\mathrm{d}x$ is equal to the moment $M_\lambda = \int_0^\infty x^\lambda n(x,t)\mathrm{d}x$ of the original distribution. Thus, (A.3) is a transformation that preserves the numerical value of the λ-moment. To confirm that the moment of the transformed distribution is not only constant but equal to the moment of the original distribution, we take the moment and obtain

$$M_{r,\lambda} = \int_0^\infty x^\lambda n_r(x,t)\,\mathrm{d}x = \int_0^\infty x^\lambda r^{1+\lambda} n_r[rx, r^{-\alpha}(t+t_0)]\,\mathrm{d}x.$$

With $y = rx$ we have

$$M_{r,\lambda} = \int_0^\infty y^\lambda n[y, r^{-\alpha}(t+t_0)]\,\mathrm{d}y = M_\lambda(t+t_0) = M_\lambda. \tag{A.4}$$

If $n(x,t)$ is a solution of the fragmentation equation the distribution $n_r(x,t)$ generated by the scale transformation will generally be a different solution. However, there is the possibility that there is a distribution that is unchanged by the transformation, i.e. is invariant under the scale transformation. When it exists this distribution is the similarity solution of the fragmentation equation and it has the same scale invariance as the equation.

To derive the form of the solution we have only to find the function form that is invariant under the scale transformation, i.e. $n_r(x,t) = n(x,t)$. If the transformed distribution is invariant it is independent of r. If $n_r(x,t) = r^{1+\lambda} n[rx, r^{-\alpha}(t+t_0)]$ is independent of r then

$$\frac{\mathrm{d}n_r(x,t)}{\mathrm{d}r} = (1+\lambda) r^\lambda n(u,w) + r^\lambda u \frac{\partial n(u,w)}{\partial u} - r^\lambda \alpha w \frac{\partial n(u,w)}{\partial w} = 0. \tag{A.5}$$

The general solution of (A.5) is

$$n(x,t) = \frac{\mu_\lambda M_{\lambda-1}^{\lambda+1}}{\mu_{\lambda-1} M_\lambda} \phi\left(\frac{\mu_\lambda M_{\lambda-1}}{\mu_{\lambda-1} M_\lambda}\right) \tag{A.6}$$

which may be confirmed by substitution into (A.5). We impose the normalization's $\mu_{\lambda-1} = 1$, $\mu_\lambda = 1$ in (A.6) and obtain the similarity form (3.1).

One can see at this point a certain generality in the construction of the similarity form. The only properties of the fragmentation equation that were used were that a moment M_λ is conserved and that the equation is invariant under the scale transformation and time translation. Nowhere was the linearity of the equation used or was it necessary to say anything about the fragmentation kernel, except that certain regularity conditions are understood. Thus, the above derivation applies as well to the quadratically nonlinear Smoluchowski coagulation equation [17] when the coagulation terms are homogeneous in the particle sizes.

Appendix B. The solution for the reduced moments

The positive zero λ and the negative zeros $\lambda_1, \lambda_2 \ldots, \lambda_p$ are solutions of

$$\frac{b_0}{k+\gamma+1} - \frac{b_1}{k+\gamma+2} - \cdots - \frac{b_p}{k+\gamma+1+p} = 1. \tag{B.1}$$

For A_k given by (3.16) we obtain

$$A_{\lambda-1} A_{\lambda-1+\alpha} \ldots A_{\lambda-1+(n-1)\alpha} = \frac{1}{\alpha^n} \frac{((\lambda_1+\lambda-1)/\alpha)_n ((\lambda_2+\lambda-1)/\alpha)_n \ldots ((\lambda_p+\lambda-1)\alpha)_n}{((\gamma+\lambda)/\alpha)_n ((\gamma+\lambda+1)/\alpha)_n \ldots ((\gamma+\lambda+p)/\alpha)_n} \tag{B.2}$$

where $(a)_n = \rho(a+1)\ldots(a+n-1)$ is the Pochhammer factorial, $(a)_n = \Gamma(a+n)/\Gamma(a)$, and Γ is the gamma function. We may write (B.2) in terms of the gamma functions as

$$\begin{aligned}
&A_{\lambda-1} A_{\lambda-1+\alpha} \ldots A_{\lambda-1+(n-1)\alpha} \\
&\quad = \frac{1}{\alpha^n} [\Gamma((\lambda_1+\lambda-1)/\alpha+n)\Gamma((\lambda_2+\lambda-1)/\alpha+n)\ldots \\
&\qquad \ldots\Gamma((\lambda_p+\lambda-1)/\alpha+n)\Gamma((\gamma+\lambda)/\alpha)\Gamma((\gamma+\lambda+1)/\alpha)\ldots \\
&\qquad \ldots\Gamma((\gamma+\lambda+p)/\alpha)] \\
&\qquad \times[\Gamma((\lambda_1+\lambda-1)/\alpha)\Gamma((\lambda_2+\lambda-1)/\alpha)\ldots \\
&\qquad \ldots\Gamma((\lambda_p+\lambda-1)/\alpha)\Gamma((\gamma+\lambda)/\alpha+n)\Gamma((\gamma+\lambda+1)/\alpha+n)\ldots \\
&\qquad \ldots\Gamma((\gamma+\lambda+p)/\alpha+n)]^{-1}
\end{aligned} \tag{B.3}$$

Substitution of (B.3) into (3.16) yields (3.18) in the text which is

$$\begin{aligned}
\mu_{\lambda-1+k} = {} & D_p(\alpha,\gamma,\lambda)\beta^k[\Gamma((k+\gamma+\lambda)/\alpha)\Gamma((k+\gamma+\lambda+1)/\alpha)\ldots \\
& \ldots\Gamma((k+\gamma+\lambda+p)/\alpha)] \\
& \times[\Gamma((k+\lambda_1+\lambda-1)/\alpha)\Gamma((k+\lambda_2+\lambda-1)/\alpha)\ldots \\
& \ldots\Gamma((k+\lambda_p+\lambda-1)/\alpha)]^{-1}
\end{aligned} \tag{B.4}$$

where $D_p(\alpha,\gamma,\lambda)$ is given by (3.19).

Appendix C. Representations of the G-function

We give here representations of the G-function for small and large η.

C.1. Power series at $\eta = 0$ and the $\eta \to 0$ limit

If $1/\alpha$ is not an integer then the poles of the gamma functions are simple and summing over the residues yields the sum of ${}_pF_p$ series given by

$$G^{p+1,0}_{p,p+1}(a_p; 0, c_p; \eta) = \frac{\Pi^p_{i=1}\Gamma(c_i)}{\Pi^p_{i=1}\Gamma(a_i)} \times_p F_p(1 - a_p; 1 - c_p; -\eta)$$
$$+ \sum_{j=1}^{p} \frac{\Pi^p_{i=1}\Gamma(c_i - c^*_j)\eta^{j/a}}{\Pi^p_{i=1}\Gamma(((\lambda_i - \gamma - 1)/\alpha) - (j/\alpha))} {}_pF_p(1 + c_j - a_p; 1 + c_j - c^*_p; -\eta) \tag{C.1}$$

where

$$a_p = \frac{\lambda_1 - \gamma - 1}{\alpha}, \frac{\lambda_2 - \gamma - 1}{\alpha}, \ldots, \frac{\lambda_p - \gamma - 1}{\alpha} \qquad c_p = \frac{1}{\alpha}, \frac{2}{\alpha}, \ldots, \frac{p}{\alpha}$$

and

$${}_pF_p(a_p; c_p; \eta) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n}{(c_1)_n(c_2)_n \ldots (c_p)_n n!}\eta^n$$

is a confluent, generalized hypergeometric series. The notation c^*_j, c^*_p in (C.1) means that the term with $i = j$ is omitted from the product and the term with the jth compound of c_p is omitted from the argument of ${}_pF_p$. For small η the leading terms are

$$G^{p+1,0}_{p,p+1}\left(\frac{\lambda_p - \gamma - 1}{\alpha}; 0, \frac{p}{\alpha}; \eta\right) = \pi_0(1 - \rho\eta + \cdots) + \pi_1\eta^{1/\alpha}(1 - \sigma\eta + \cdots) + \cdots \tag{C.2}$$

where $\pi_0, \pi_1, \rho, \sigma$ are constants. Thus, according to (4.4),

$$\lim_{n\to 0} \overline{\phi}(\eta) = C_0\eta^{\gamma/\alpha} \tag{C.3}$$

when $\eta = z^\alpha/\beta^\alpha$ and C_0 is a constant which can be obtained from (C.1) and (4.4).

C.2. An integral representation and the $\eta \to \infty$ limit

It is shown in [15] that the G-function can be represented as

$$G^{p+1,0}_{p,p+1}\left(\frac{\lambda_p - \gamma - 1}{\alpha}; 0, \frac{p}{\alpha}; \eta\right)$$
$$= \frac{1}{\Gamma((\lambda_1 - \gamma - 2)/\alpha)\Gamma((\lambda_2 - \gamma - 3)/\alpha) \ldots \Gamma((\lambda_p - \gamma - 1 - p)/\alpha)}$$
$$\times \int_0^\infty \int_0^\infty \ldots \int_0^\infty \mathrm{d}u_1\,\mathrm{d}u_2 \ldots \mathrm{d}u_p$$
$$\times u_1^{((\lambda_1-\gamma-2)/\alpha)-1}(1 + u_1)^{-((\lambda_1-\gamma-1)/\alpha)} u_2^{((\lambda_2-\gamma-3)/\alpha)-1}(1 + u_2)^{-((\lambda_2-\gamma-1)/\alpha)} \ldots$$
$$\ldots u_p^{((\lambda_p-\gamma-1-p)/\alpha)-1}(1 + u_p)^{-((\lambda_p-\gamma-1)/\alpha)} \exp[-\eta(1 + u_1)(1 + u_2) \ldots$$
$$\ldots (1 + u_p)]. \tag{C.4}$$

From this representation one can see by inspection that the integrals are finite if we have the following lower bounds on the zeros of A_k,

$$\lambda_1 > \gamma + 2, \lambda_2 > \gamma + 3, \ldots, \lambda_p > \gamma + 1 + p. \tag{C.5}$$

The expansion of $G^{p+1,0}_{p,p+1}(\eta)$ in powers of η^{-1} can be obtained from (C.4). Following the same steps given in [15] for $\lambda = 1$ we obtain from (C.4) the limit

$$\lim_{\eta\to\infty} \bar{\phi}(\eta) \sim \frac{\alpha}{\beta^{\lambda}} \eta^{(\gamma+\Lambda)/\alpha} \exp(-\eta) \tag{C.6}$$

where

$$\Lambda = -\sum_{j=1}^{p} (\lambda_j - \gamma - 1 - j). \tag{C.7}$$

For $p = 0$ we see from the definition of the G-function that $G^{1,0}_{0,1}(\eta) = \exp(-\eta)$. According to Luke [18], for $p \geqslant 1$ the expansions of $G^{p+1,0}_{p,p+1}(\eta)$ in powers of η^{-1} are asymptotic expansions. Thus, for $p \geqslant 1$ (C.6) is the limit of an asymptotic expansion.

For example, from (C.4) for $p = 1$ we have

$$G^{2,0}_{1,2}\left(\frac{\lambda_1 - \gamma - 1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right) = \exp(-\eta) \frac{1}{\Gamma((\lambda_1 - \gamma - 2)/\alpha)} \times \int_0^\infty \mathrm{d}u\, u^{((\lambda_1-\gamma-2)/\alpha)-1} (1+u)^{-((\lambda_1-\gamma-1)/\alpha)} \exp[-\eta(1+u)].$$

We expand $(1+u)^{-((\lambda_1-\gamma-1)/\alpha)}$ in a power series at $u = 0$, change to the variable $\xi = \eta u$ and obtain

$$G^{2,0}_{1,2}\left(\frac{\lambda_1 - \gamma - 1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right) = \eta^{-(\lambda-\gamma-2)/\alpha} \exp(-\eta) \frac{1}{\Gamma(\lambda_1 - \gamma - 2)/\alpha)} \times \int_0^\infty \mathrm{d}\xi\, \xi^{((\lambda_1-\gamma-2)/\alpha)-1} \exp(-\xi) \left(1 - \frac{1}{1!}\left(\frac{\lambda_1-\gamma-1}{\alpha}\right)\xi\eta^{-1} + \cdots \cdots + \frac{1}{n!}\left(\frac{\lambda_1-\gamma-1}{\alpha}\right)_n \xi^n \eta^{-n}\right).$$

Integrating term by term gives the series

$$G^{2,0}_{1,2}\left(\frac{\lambda_1 - \gamma - 1}{\alpha}; 0, \frac{1}{\alpha}; \eta\right) \sim \eta^{-(\lambda_1-\gamma-2)/\alpha} \exp(-\eta)\, {}_2F_0\left(\frac{\lambda_1-\gamma-2}{\alpha}, \frac{\lambda_1-\gamma-1}{\alpha}; ; -\eta^{-1}\right) \tag{C.8}$$

where the blank entry in the argument means there is no Pochhammer factorial in the denominator and in the denominator we used

$$\Gamma\left(\frac{\lambda_1-\gamma-2}{\alpha} + n\right) = \Gamma\left(\frac{\lambda_1-\gamma-2}{\alpha}\right)\left(\frac{\lambda_1-\gamma-2}{\alpha}\right)_n.$$

Since there are two Pochhammer factorials in the numerator and no Pochhammer factorial in the denominator it is evident that the series diverges for all η, but it converges asymptotically. The leading terms for the asymptotic expansion for $p = 2$ and $\lambda = 1$ are shown in [15]. The complexity of the expansion increases very fast as p increases.

Appendix D. The recursion equation for the reduced moments

Multiplying (4.9) by z^k and integrating gives

$$(-k + \lambda)\mu_k = \frac{\alpha}{\beta^{\alpha}}\left[-\mu_{k+\alpha} + \int_0^\infty \mathrm{d}z\, z^k \int_z^\infty b\left(\frac{z}{w}\right) w^{\alpha-1}\phi(w)\, \mathrm{d}w\right] \tag{D.1}$$

where we have used the boundary conditions $z^{k+1}\phi(z)|_{z=0} = 0$, $z^{k+1}\phi(z)|_{z=\infty} = 0$ and $\mu_k = \int_0^\infty z^k\phi(z)\,\mathrm{d}z$. Changing the way we do the double integration we obtain

$$\int_0^\infty \mathrm{d}z\, z^k \int_z^\infty b(z/w)w^{\alpha-1}\phi(w)\,\mathrm{d}w = \int_0^\infty \mathrm{d}w\left(\int_0^w \mathrm{d}z\, z^k b(z/w)\right)\phi(w)\,\mathrm{d}w. \tag{D.2}$$

Then with the change of variable $w = w, r = z/w$ we have

$$\int_0^\infty dw\left(\int_0^w \mathrm{d}z\, z^k b(z/w)\right)\phi(w)\,\mathrm{d}w = \int_0^1 r^k b(r)\,\mathrm{d}r\,\mu_{k+\alpha}. \tag{D.3}$$

With (D.2) and (D.3) in (D.1) we obtain

$$(-k+\lambda)\mu_k = \frac{\alpha}{\beta^\alpha}\left(-1+\int_0^1 r^k b(r)\,\mathrm{d}r\right)\mu_{k+\alpha}. \tag{D.4}$$

With the definition of A_k given by (2.4), we see that (D.4) is the recursion equation (3.12) derived in another way in the text.

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